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On the compatibility of relativistic wave equations in Riemann spaces: II

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Abstract. Some details of a previous paper relating to the internal consistency in a V_4 of spin S equations ($S \geqslant \frac{3}{2}$) are re-examined and a special case is brought to light. This occurs when one of the field spinors has indices of only one kind. It is argued that the prescription of minimal gravitational coupling is to be abandoned. In the special case just referred to, the minimally coupled equations are modified by the addition to one of them of a certain term which contains the Weyl tensor as a factor, so that consistency in an arbitrary V_4 then obtains.

1. Introduction

Some years ago I investigated (Buchdahl 1962a, hereafter referred to as I) the compatibility, i.e. self-consistency, of pairs of relativistic spin S ($=: \frac{1}{2}n$), non-zero mass equations

$$p^{\dot{\mu}_{s+1}}{}_{\nu_t}\xi^{\dot{\mu}_1\cdots\dot{\mu}_s\nu_1\cdots\nu_t} = \kappa\eta^{\dot{\mu}_1\cdots\dot{\mu}_{s+1}\nu_1\cdots\nu_{t-1}} \tag{1.1a}$$

$$p_{\dot{\mu}_{s+1}}{}^{\nu_t}\eta^{\dot{\mu}_1\cdots\dot{\mu}_{s+1}\nu_1\cdots\nu_{t-1}} = \kappa\xi^{\dot{\mu}_1\cdots\dot{\mu}_s\nu_1\cdots\nu_t} \tag{1.1b}$$

in a Riemannian spacetime V_4 of signature -2. Here s and t are non-negative integers such that $s+t=n\geqslant 3$; whilst if X is any spin tensor (indices suppressed) $p^{\dot{\mu}}{}_{\nu}X$ stands for the transvection $\sigma^{k\dot{\mu}}{}_{\nu}X_{;k}$ of the Pauli tensor–spinor with the covariant derivative of X. These first-order equations result from the corresponding flat space equations merely by the formal replacement of partial by covariant derivatives, the curvature tensor itself not being brought in: one may speak of minimal gravitational coupling. The main conclusion reached in I was this: the pair (1.1) is internally consistent if and only if the V_4 has constant Riemannian curvature when $S>\frac{3}{2}$ or is an Einstein space when $S=\frac{3}{2}$.

I now return to the earlier work for the following reasons. First, a hiatus present in the establishment of the result just quoted (when $S>\frac{3}{2}$) needs to be removed. This is done in § 2. Second, the recognition that one has a special case whenever $t=n$ previously slipped through the net. The consequences of this oversight are examined in § 3: it turns out that the V_4 is, in this case, required to be merely conformally flat. Third, it is argued in § 4 that in the present context the case for the mandatory adoption of minimal coupling is not compelling. One should therefore contemplate appropriate modifications of (1.1) for which self-consistency obtains in an *arbitrary* V_4. This endeavour is more likely to meet with success in the special case $t=n$ just referred to since the constraint imposed upon (1.1) by the requirement of mutual compatibility is then weaker than for other values of t. A self-consistent set of equations is, indeed, constructed in § 5. From this the wave equation obeyed by ξ is derived explicitly in § 6.

2. Previous work revisited

Using the abbreviated notation introduced in § 2 of I the equations (1.1) read

$$p^{\dot\mu_{s+1}}{}_{\nu_t}\xi = \kappa\eta \qquad p_{\dot\mu_{s+1}}{}^{\nu_t}\eta = \kappa\xi. \tag{2.1a, b}$$

By transvection with $\gamma_{\dot\mu_{s+1}\dot\mu_s}$ and $\gamma_{\nu_{t-1}\nu_t}$ respectively there follow the 'subsidiary conditions'

$$p_{\dot\mu_s\nu_t}\xi = 0 \qquad p_{\dot\mu_{s+1}\nu_{t-1}}\eta = 0. \tag{2.2a, b}$$

In outline, the argument presented in I proceeded along the following lines. Alternatively eliminating ξ and η between (2.1a) and (2.1b) and again taking the symmetries of ξ and η into account one obtains the equations of constraint

$$S^{kl}{}_{\nu_{t-1}\nu_t}\xi_{;[kl]} = 0 \qquad S^{kl}{}_{\dot\mu_s\dot\mu_{s+1}}\eta_{;[kl]} = 0. \tag{2.3a, b}$$

These also follow directly from (2.2a) and (2.2b) if use is made of (2.1b) and (2.1a), respectively. Contemplating only (2.3a), and requiring that at any point the initial values of the field amplitudes be freely assignable, this yields the condition I (19), i.e.

$$\sum_{r=1}^{\dot s} \sigma^{k\dot\mu_r}{}_{\nu_{t-1}}\sigma^{l}{}_{\dot\lambda\nu_t}\bar\Delta^{(\lambda)_r}\Delta E_{kl} + \tfrac{1}{2}\sum_{r=1}^{t-2} S^{kl}{}_{\nu_{t-1}\nu_t}S^{mn\nu_r}{}_{\lambda}\bar\Delta\Delta^{(\lambda)_r}C_{klmn} = 0. \tag{2.4}$$

(Recall that in I the vector ϕ_k was taken to vanish.) Contracting over all free dotted indices the contraction of the first term on the left vanishes identically. By subsequent contraction over $t-3$ pairs of free undotted indices one arrives in effect at the condition $C_{klmn} = 0$. It is at this point that a gap was inadvertently left in § 3 of I, for one must now return to (2.4) in which only the first sum survives. Contract this time over all free undotted and all but one pair of free dotted indices. As a result one is left with the condition of the form $\sigma^{k\dot\alpha}{}_{\beta\alpha}{}^{l}{}_{\dot\gamma\delta}E_{kl} = 0$ which entails that E_{kl} must vanish. Compatibility thus requires the V_4 to be a conformally flat Einstein space and therefore a space S_4 of constant Riemannian curvature; see, however § 3. On the other hand (2.3b) is then also satisfied so that the mutual compatiblity of (2.1a, b) is assured in an S_4.

3. The special case $t = n$

The unqualified conclusion that the V_4 must be an S_4 if the equations (2.1) are to be mutually compatible has to be treated with caution. It is justified only if (2.3) requires both C_{klmn} and E_{kl} to vanish. When $S = \frac{3}{2}$ and $t = 2$, however, only the first term on the right of (2.4) survives, so that this condition merely requires E_{kl} to vanish. When it does so (2.36) is also satisfied. In this case, then, the V_4 must be an Einstein space, a result found long ago (Buchdahl 1958).

A more far reaching special case arises when $t = n$ for *all* values of $n \geqslant 3$. (Recall from § 2 of I that one can always take $t \geqslant 3$, a convention in harmony with (2.4).) When $t = n$ the absence of any dotted indices from ξ entails the absence of the first term on the left of (2.4). Indeed, since η has only a single dotted index one has merely the single subsidiary condition

$$p_{\dot{\mu}_1 \nu_n}\eta = 0 \tag{3.1}$$

and therefore only the *one* equation of constraint

$$S^{kl}{}_{\nu_{n-1}\nu_n}\xi_{;kl} = 0. \tag{3.2}$$

This in turn leads to (2.4) with the first sum on the left absent from it. Bearing the work of § 3 of I in mind, it follows that *when $t = n$ the equations (2.1a, b) are mutually compatible if and only if the V_4 is conformally flat.*

4. Remarks on minimal coupling

The equations (2.1*a*, *b*) are minimally coupled (to gravitation). Formally this means that they result from the corresponding flat space equations merely by replacing in the latter all partial by covariant derivatives. In other words, the Riemann tensor is not allowed to appear explicitly in these *first-order* equations. If one omits to specify that the minimal coupling prescription is intended to apply to first-order equations one meets an ambiguity in the sense that second-order equations derived by iteration from minimally coupled first-order equations are not, in general, themselves minimally coupled. (For example, the curvature scalar R appears explicitly in the iterated second-order Dirac equation.)

Now, the minimal coupling prescription applied to the spin S equations generates equations which are mutually incompatible in an arbitrary V_4. Under these circumstances one may question the legitimacy of the insistence on minimal coupling. Moreover, one does well to reflect here on the fact that the equations of motion of a *classical* spinning particle are not minimally coupled. In any event, the requirement of minimal coupling seems to be little more than a reflection of an ill-defined appeal to 'simplicity'. If so, in a general V_4 the 'simplest' equations corresponding to the usual flat space equations will not be (2.1), for they are mutually incompatible, but rather the 'simplest possible'—whatever that may mean—pair of mutually compatible equations which in flat space reduce to the usual flat space equations. They can always be thought of as modifications of (2.4) in which further terms are added to their right-hand members, granted that the additional terms depend linearly on ξ and η and vanish when $R_{klmn} = 0$.

It is not at all obvious how the required modifications might be achieved, if they be possible at all. Furthermore, in flat space, corresponding to different choices of the value of t one has for given S alternative pairs of equations (2.1) and all these are mutually equivalent (cf I § 5). In a general V_4, granted minimal coupling, it no longer makes sense to contemplate this mutual equivalence on account of the internal inconsistency of the individual pairs. In any event, formally one is no longer dealing with 'free-field equations' and in principle one has to think of different gravitational couplings for the various possible values of t. In particular, then, it may be possible to achieve compatibility for one value of t rather than another.

In the light of these remarks I confine my attention here to the case $t = n$, for then the unmodified equations exhibit—very loosely speaking—the least degree of incompatibility in the sense that one has only (3.2) to contend with, there being no second condition of the kind (2.3*b*). In these circumstances, the problem of finding the required modifications of (2.1) may be expected to be more tractable than when $t \neq n$. This, indeed, turns out to be the case.

5. Compatible equations when $t = n$

When $t = n$ the equations (2.1) are

$$p^{\dot{\mu}_1}{}_{\nu_n}\xi = \kappa\eta \qquad p_{\dot{\mu}_1}{}^{\nu_n}\eta = \kappa\xi. \tag{5.1a, b}$$

Their mutual incompatibility may be thought of as coming about by the failure, in general, of the left-hand member of (5.1b) to be symmetric in ν_{n-1} and ν_n; for this is reflected by (3.1) and in turn by (3.2). It suggests itself to allow for this lack of symmetry by the addition on the right of (5.1b) of an appropriate spinor $\hat{\beta}^{\nu_1\ldots\nu_n}$ (=: $\hat{\beta}$), symmetric in its first $n-1$ superscripts. Since one is effectively only concerned with its transvection with $\gamma_{\nu_{n-1}\nu_n}$ it is natural to make the ansatz

$$\hat{\beta} = [(n-1)/n]\beta^{(\nu_1\ldots\nu_{n-2}}\gamma^{\nu_{n-1})\nu_n} \tag{5.2}$$

where $\beta^{\nu_1\ldots\nu_{n-2}}$ (=: β) is some symmetric spinor. Note that

$$\gamma_{\nu_{n-1}\nu_n}\hat{\beta} = \beta. \tag{5.3}$$

Transvection of the modified version of (5.1) throughout with $\gamma_{\nu_{n-1}\nu_n}$ shows that

$$p_{\dot{\mu}_1\nu_{n-1}}\eta = -\beta. \tag{5.4}$$

Eliminating η between (5.1a) and (5.4) one finds immediately that

$$\kappa\beta = -S^{kl}{}_{\nu_{n-1}\nu_n}\xi_{;kl}. \tag{5.5}$$

It is now manifest that the condition (3.2) is a result of prescribing from the outset that β be zero; but, as has been argued in § 4, it is not mandatory to do this.

Since $S^{(kl)}{}_{\alpha\beta} = 0$ only the skew symmetric part of $\xi_{;kl}$ occurs in (5.5) which may therefore be written

$$2\kappa\beta = S^{kl}{}_{\nu_{n-1}\nu_n}\sum_{r=1}^{n} P^{\nu_r}{}_{\lambda kl}\xi^{(\lambda)_r}. \tag{5.6}$$

Taking the identities (5.4) and (6.3) of Buchdahl (1962b) into account, the terms which have $r = n-1$ and $r = n$ vanish, whilst the remaining terms all contain the Weyl tensor as a factor, so that

$$\beta = \tfrac{1}{4}\kappa^{-1}C_{klmn}S^{kl}{}_{\nu_{n-1}\nu_n}\sum_{r=1}^{n-2} S^{mn\nu_r}{}_{\lambda}\xi^{(\lambda)_r}. \tag{5.7}$$

Greater formal simplicity is attained by defining the 'Weyl spinor'

$$C_{\alpha\beta\gamma\delta} := \tfrac{1}{4}S^{kl}{}_{\alpha\beta}S^{mn}{}_{\gamma\delta}C_{klmn} \tag{5.8}$$

for then (5.7) reads

$$\beta = \kappa^{-1}\sum_{r=1}^{n-2} C_{\nu_{n-1}\nu_n\lambda}{}^{\nu_r}\xi^{(\lambda)_r} = (n-2)\kappa^{-1}C_{\alpha\beta\gamma}{}^{(\nu_1}\xi^{\nu_2\ldots\nu_{n-2})\alpha\beta\gamma}. \tag{5.9}$$

This explicit expression for β may now be used on the right of the modified equation (5.1*b*), i.e. $p_{\dot{\mu}_1}{}^{\nu_n}\eta = \kappa\xi + \hat{\beta}$. In short, the particular spin S equations

$$p^{\dot{\mu}_1}{}_{\nu_n}\xi = \kappa\eta \tag{5.10a}$$

$$p_{\dot{\mu}_1}{}^{\nu_n}\eta = \kappa\xi - \kappa^{-1}[(n-1)(n-2)/n]\gamma^{\nu_n(\nu_1}C_{\alpha\beta\gamma}{}^{\nu_2}\xi^{\nu_3\ldots\nu_{n-1})\alpha\beta\gamma} \tag{5.10b}$$

are mutually compatible in an arbitrary V_4. In principle, therefore, it is now no longer necessary to ignore the effect of the ξ,η field upon the gravitational field.

6. Second-order equation for ξ

If η be eliminated between (2.1*a*) and (2.1*b*) the resulting equation for ξ contains an additive term not symmetric in $\nu_1, \ldots, \nu_n$. This asymmetry will be expected to be just accounted for by the supplementary term in (5.10*b*). That this is the case may be confirmed by detailed calculation. In short, recalling I(6), ξ obeys the equation

$$(\Box + 2\kappa^2)\xi = 2S^{kl}{}_{\lambda}{}^{(\nu_n}\xi^{\nu_1\ldots\nu_{n-1})\lambda}{}_{;kl}. \tag{6.1}$$

The right-hand member may be dealt with in the manner by now familiar and one ends up with the desired equation

$$[\Box + 2\kappa^2 + \tfrac{1}{12}(n+2)R]\xi^{\nu_1\ldots\nu_n} = -2(n-1)C_{\alpha\beta}{}^{(\nu_1\nu_2}\xi^{\nu_3\ldots\nu_n)\alpha\beta} \tag{6.2}$$

in harmony with I(2*b*). Given a solution of this equation, η is then found from (5.10*a*).

References

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